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Lyapunov-Designed MRAS-Linearized Error Equation Results

B. K. Colburn*
Texas A&M University, College Station, Tex.
and
J. S. Boland, III†
Auburn University, Auburn, Alabama

Introduction

THE large volume of work in the field of model-reference adaptive control (MRAS) during the last decade has resulted in a large number of adaptive controller/observer techniques.[1,2] For those methods where asymptotic stability has been proved, it has been due to the application of Lyapunov's Second Method because of the inherent nonlinear and complex nature of these equations. The nonlinearities cause great difficulty in analyzing such adaptive systems. A concept that has met with some success in the analyzing MRAS systems is that of the linearized error characteristic equation (LECE) approach.[3-5] The purpose of this Note is to present some conjectured propositions based on results of applying the LECE concept to some published adaptive techniques. These potential theorems, in addition to being of interest in their own right, can provide insight into design and analysis of MRAS controller/observer systems. Additionally, they provide a means of demonstrating the manner in which results from Lyapunov Theory agree with classical linear stability concepts when comparing with the linearized root locus expressions using the LECE approach.

Linearized Error Equation Approach

The LECE technique is a general approach for obtaining root locus expressions for systems involving adaptation, whether an observer or controller. It relates adaptation design parameters to the adaptive error transient response. The approach consists of deriving a set of linearized error equations for each adaptive method about an operating point and manipulating these truncated linear equations into the form $1+KG(s)$ for plotting root loci. For the case of an adaptive control system, the plant is adapted to track a reference model, and for the observer the plant is fixed and an adjustable model is adapted.

The LECE concept, based on error equation analysis, has a strong theoretical basis in the case of the adaptive control, but requires clarification in the observer (or identification) case. This is because the Lyapunov design theory for observers generally requires a "frequency richness" condition on the input to insure parameter identification. In developing the LECE for observers, it is assumed constant inputs are applied. This effect can be essentially met as long as any a.c. variation on u, and their derivatives, are "small" with respect to any bias terms on u. Without the assumption of constant u, meaningful LECE results do not appear possible.

Applying the LECE technique to five published adaptation methods, one obtains a series of LECE expressions which can be used for design and analysis purposes. The details are given in Table 1.[4] Methods 1, 2, 4, and 5 are adaptive control laws, method 1 being the Winsor and Roy,[6] method 2 the Ten Cate,[7] method 4 the Gilbart et al.,[3] and method 5 the Sutherlin and Boland method.[8] Method 3 is the adaptive observer of Kudva and Narendra.[9,10]

Common to methods 1, 2, 4, and 5 is the Z factor (Table 1), which includes weighting constants q_{ij}, elements of a positive definite (p.d.) symmetric matrix Q satisfying

$$A_m^T Q + QA_m = -C \tag{1}$$

where A_m is a stable model matrix that is to be tracked and C is a p.d. symmetric matrix. Method 3 employs a weighting constant α, $0<\alpha<2$, P is a p.d. symmetric matrix, and $\lambda_{\max}$ is the largest eigenvalue of P.

LECE Conjectures

Using the LECE approach on the previous methods, the resulting root locus expressions are given in Table 1. From these expressions, and some additional material, five conjectured propositions can be formulated.

Conjecture 1. If A_m is a constant, stable $n\times n$ matrix in the phase variable form, C is a p.d. symmetric $n\times n$ matrix, Q is a p.d. symmetric matrix satisfying Eq. (1), and q_{ij} is the ijth element of Q, then the roots x_i, $i=1,2,\ldots, n$ of

$$\sum_{i=1}^{n} q_{ni}x^{i-1}=0 \tag{2}$$

satisfy $R_e\{x_i\}<0, \quad i=1,2,\ldots,n.$

This particular conjecture comes about from an examination of the LECE's of methods 1, 4, and 5 of Table 1. The roots x_i represent the location of open-loop zeros of a root locus expression. Since the root locus gain can become infinite, if the LECE approach is to yield realistic results, the zeros would have to be in the open left-hand s plane for the adaptive system to be stable for large values of inputs.

From a study of the LECE of method 2 in Table 1, the following proposition can be formulated.

Conjecture 2. If A_m is a constant, stable $n\times n$ matrix in the phase-variable form for the single-input system $\dot{x}=A_m x+bu$, where $b^T=(00\ldots01)$, C is a p.d. symmetric $n\times n$ matrix, γ is a nonnegative scalar, Q is a p.d. symmetric $n\times n$ matrix satisfying Eq. (1) and q_{ij} is the ijth element of Q, then the roots x_i, $i=1,2,\ldots, n$ of

$$\sum_{i=1}^{n} [q_{ni}x^{i-1}]+\gamma\,|xI-A_m|=0 \tag{3}$$

are such that $Re\{x_i\}<0$.

Adaptive observers designed by Lyapunov Theory also exhibit interesting stability properties, as given in the following proposition.

Conjecture 3. If α is a scalar such that $0<\alpha<2$, P is a p.d. symmetric $n\times n$ matrix, and $\lambda_{\max}$ is the largest eigenvalue of P, then the roots z_i, $i=1,2,\ldots,(n-1)$ of

$$(z-1)z^{n-2}+\frac{\alpha}{\lambda_{\max}}\sum_{j=2}^{n}p_{jn}z^{n-j}=0 \tag{4}$$

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*Assistant Professor, Department of Electrical Engineering. Member AIAA.

†Associate Professor, Department of Electrical Engineering. Member AIAA.

Table 1 Linearized error equations of the form $1+KG(s)=0$[a]

Method	Adaptive gain	LECE
1 Windsor and Roy[6]	$\alpha_{ij}\int_{t_0}^{t} Z\,dt$	$1+\dfrac{k\left[\sum_{j=1}^{n} q_{jn}s^{j-1}\right]}{s\Delta_m(s)}=0$
2 Ten Cate[7]	$\alpha_{ij}\int [Z+\gamma_j f(\delta)]\,dt$	$1+\dfrac{k\left[\sum_{j=1}^{n} q_{jn}s^{j-1}+\gamma\Delta_m(s)\right]}{s\Delta_m(s)}=0$
3 Kudva and Narendra	$K_{ij}(k)-\dfrac{\gamma Y}{R}$	$1+\dfrac{k\left[\sum_{j=2}^{n} p_{jn}Z^{n-j}\right]}{(z-1)Z^{n-2}}=0$
4. Gilbart et al.[3]	$\alpha_{ij}\int_{t_0}^{t} Zdt+\beta_{ij}Z$	$1+\dfrac{\left[K_1+K_2 s\right]\left[\sum_{j=1}^{n} q_{jn}s^{j-1}\right]}{S\Delta_m(s)}=0$
5. Sutherlin and Boland[8]	$\alpha_{ij}\int Zdt+\beta_{ij}Z$ $+\rho_{ij}\dfrac{d}{dt}Z$	$1+\dfrac{\left[K_1+K_2 s+K_3 s^2\right]\left[\sum_{j=1}^{n} q_{jn}s^{j-1}\right]}{s\Delta_m(s)}=0$

[a] $Z=\sum_{k=1}^{n} e_k q_{ki} x_{p_j}$; $Y=Pe(k)x_p^T(k-1)$; $\Delta_m(s)=|sI-A_m|$; $R=x_p^T(k-1)x_p(k-1)+u(k-1)^2$; $f(\delta)$ is the system parameter coefficient misalignment, where $\alpha>0$, β and $\rho\geq 0$; K_1, K_2, $K_3=f$ (plant states, inputs, α, β, ρ, etc.)

satisfy the inequality

$$|z_i|<1 \qquad \forall i, \quad i=1,2,\ldots,(n-1)$$

This conjecture comes about from an examination of the LECE of method 3 in Table 1.

The open-loop zeros of the LECE of the method of Ref. 3 are:

$$\sum_{j=2}^{n} p_{jn} z^{n-j}=0 \tag{5}$$

and the root locus gain k is

$$k=\frac{\alpha p_{nn}\left[\sum_{i}^{\ell} x_{p_i}^{0^2}+u^{0^2}\right]}{\left[\sum_{i=1}^{n} x_{p_i}^{0^2}+u^{0^2}\right]\lambda_{\max}} \tag{6}$$

where $\ell\leq n$, and Σ_i represents a sum of i terms not necessarily in consecutive order. This accounts for the fact that if not all the plant terms are identified (they may be known in advance), then the root locus gain is appropriately modified. It should be noted that the zero compensator terms given by Eq. (5) are such that the zeros need not be inside the unit circle. The reason for this startling result comes from Eq. (6), where the maximum value of the root locus gain is limited to a maximum magnitude

$$|k|_{\max}=\alpha p_{nn}/\lambda_{\max} \tag{7}$$

where α, $\lambda_{\max}$, and p_{ij} were defined previously. The reason for this "gain limiting," unique to the discretized case, comes from the Lyapunov Theory results. To insure asymptotic error stability, a division factor

$$\sum_{i=1}^{n} x_{p_i}^{0^2}+u^{0^2} \tag{8}$$

occurs. Without this term, k could increase without bound as the plant state and input values increase, as in the continuous time methods. This points to the consistency of the exact theory and the linearized analysis approach. This result also suggests a basic requirement for discrete adaptation laws, namely some form of gain limiting as a function of states and inputs is required.

Suppose a system has plant equations of the form

$$\dot{x}_p=A_p(t)x_p+B_p(t)u \tag{9}$$

and the corresponding model, which is to be used to force the plant states to track the model states, is given by

$$\dot{x}_m=A_m x_m+B_m u \tag{10}$$

where x_p, x_m are n-vectors, $A_p(t)$ and $B_p(t)$ are $n\times n$ and $n\times r$ possibly time varying matrices, u is an r-input vector, and A_m and B_m are $n\times n$ and $n\times r$ constant matrices with A_m stable. Using the Lyapunov V function developed in Ref. 8,

$$\begin{aligned}V=e^TQe&+\sum_{i,j=1}^{n}\frac{1}{\alpha_{ij}}\Big\{a_{ij}+\beta_{ij}\sum_{k=1}^{n} e_k q_{ki} x_{p_j}\\&+\rho_{ij}\frac{d}{dt}\Big[\sum_{k=1}^{n} e_k q_{ki} x_{p_j}\Big]\Big\}^2+\sum_{i,j=1}^{n}\rho_{ij}\Big[\sum_{k=1}^{n} e_k q_{ki} x_{p_j}\Big]\\&+\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{1}{\gamma_{ij}}\Big\{b_{ij}+\delta_{ij}\sum_{k=1}^{n} e_k q_{ki} u_j\\&+\sigma_{ij}\frac{d}{dt}\Big[\sum_{k=1}^{n} e_k q_{ki} u_j\Big]\Big\}^2+\sum_{i=1}^{n}\sum_{j=1}^{r}\sigma_{ij}\Big[\sum_{k=1}^{n} e_k q_{ki} u_j\Big]\end{aligned} \tag{11}$$

where $e=x_m-x_p$, $[a_{ij}]=A_m-A_p$, $[b_{ij}]=B_m-B_p$, α_{ij} and $\gamma_{ij}>0$, β_{ij},ρ_{ij}, δ_{ij}, σ_{ij}, ≥ 0, $Q=Q^T>0$, it can be shown that the resulting $\dot{V}$ function is

$$\dot{V}=\overset{①}{e^T(A_m^TQ+QA_m)e}-2\sum_{i=1}^{n}\sum_{j=1}^{n}\beta_{ij}\overset{②}{\sum_{k=1}^{n}e_k\, q_{ki}x_{p_j}}$$

$$\overset{③}{-2\sum_{i=1}^{n}\sum_{j=1}^{r}\delta_{ij}\sum_{k=1}^{n}e_kq_{ki}u_j} \tag{12}$$

if the adaptive control gain rates are selected as

$$\dot{a}_{ij}=\alpha_{ij}Z_{ij}+\beta_{ij}\frac{\mathrm{d}}{\mathrm{d}t}[Z_{ij}]+\rho_{ij}\frac{\mathrm{d}^2}{\mathrm{d}t^2}[Z_{ij}] \tag{13}$$

$$\dot{b}_{ij}=\gamma_{ij}Y_{ij}+\delta_{ij}\frac{\mathrm{d}}{\mathrm{d}t}[Y_{ij}/+\sigma_{ij}\frac{\mathrm{d}^2}{\mathrm{d}t^2}[Y_{ij}] \tag{14}$$

with

$$Z_{ij}=\sum_{k=1}^{n}e_kq_{ki}x_{p_j},\quad Y_{ij}=\sum_{k=1}^{n}e_kq_{ki}u_j \tag{15}$$

If $\rho_{ij}=\sigma_{ij}=0$, then Eq. (11) reduces to the V function in Ref. 3, but $\dot{V}$ in Eq. (12) still results from using the reduced version of Eqs. (13) and (14). Note that the last two terms in Eq. (13) are negative definite in e. It is well known that, if A_m is a stable matrix, there exists a p.d. symmetric Q matrix satisfying Eq. (1). Hence from Lyapunov Theory, the plant-model system is asymptotically stable in e. Use of Eq. (1) implies a sufficient condition for stability, the information in terms ② and ③ of Eq. (17) are ignored. Under a set of not-too-restrictive conditions, it can be shown[11] that $\dot{V}$ can be written as

$$\dot{V}=e^TWe-2\sum_{i=1}^{n}\sum_{j=1}^{n}\beta_{ij}\sum_{k=1}^{n}e_kq_{ki}x_{pj} \tag{16}$$

where

$$W=A_m^TQ+QA_m-\Omega\hat{q}\hat{q}^T \tag{17}$$

If W is negative definite (n.d.), then Eq. (12) is n.d., and if the resulting Q is p.d., then the system will be asymptotically stable in e. Under the constraint of the earlier restrictions, Eq. (17) then may be used in place of Eq. (1) for insuring asymptotic stability. Based on the preceding, the following conjecture is advanced.

Conjecture 4. If A_m is an $n\times n$ real matrix with eigenvalues λ_i and $Re[\lambda_i]<0$, $i=1,2,...n$, W is a negative-definite symmetric matrix, Ω is a positive real number, and $\hat{q}$ is the nth column of an $n\times n$ real matrix $Q=(q_1q_2...\hat{q}_n)$, then there exists a unique positive-definite symmetric $n\times n$ matrix Q satisfying Eq. (17).

Use of Eq. (17) can make it possible to obtain a wide selection of q_{ij} values in Eq. (2). If only Eq. (1) is used, then the capability of using $\dot{V}$ is ignored. Based on this result plus Conjecture 1, the following conjecture is advanced.

Conjecture 5. If A_m is a constant, stable, $n\times n$ matrix in phase-variable form, W is a p.d. symmetric matrix, Q is a p.d. symmetric $n\times n$ matrix satisfying Eq. (17), and $\hat{q}$ is the nth column of Q, then the roots x_i, $i=1,2,..., n$ of Eq. (17) satisfy $Re[x_i]<0$, $i=1,2,..., n$.

Summary

In general, the nonlinear complex nature of MRAS control laws precludes a clear analysis of a system operating under adaptive control. Using a linearization approach, a number of potential propositions resulted that are of interest in their own right in math analysis, as well as providing a means for interpreting certain adaptive control stability and design conditions. This reinforces the utility of the LECE approach in MRAS systems analysis.

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Turbulent Mixing Characteristics for a D_2/HCl Electric Discharge Gasdynamic Laser

Peter K. Wu,* Paul F. Lewis,†
and
Raymond L. Taylor‡
Physical Sciences Inc., Woburn, Mass.

Introduction

VARIOUS types of electric discharge gasdynamic lasers (EDGDL) have been under investigation at the Naval Research Laboratory.[1,2] This concept is as follows: a diatomic species, such as N_2, CO, D_2, etc., and a monatomic diluent mixture are passed through a glow discharge. A substantial fraction of the discharge energy is dissipated in exciting the vibrational mode of the diatomic molecule. The vibrationally excited gas is expanded through a supersonic

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*Principal Scientist. Member AIAA.
†Senior Scientist.
‡Principal Scientist.